Entanglement purification of Gaussian continuous variable quantum states

Lu-Ming Duan[1,2*], G. Giedke[1], J. I. Cirac[1], and P. Zoller[1]
[1] *Institut für Theoretische Physik, Universität Innsbruck, A-6020 Innsbruck, Austria*
[2] *Department of Physics, University of Science and Technology of China, Hefei 230026, China*

We describe an entanglement purification protocol to generate maximally entangled states with high efficiencies from two-mode squeezed states or from mixed Gaussian continuous entangled states. The protocol relies on a local quantum non-demolition measurement of the total excitation number of several continuous variable entangled pairs. We propose an optical scheme to do this kind of measurement using cavity enhanced cross–Kerr interactions.

PACS numbers: 03.67.Hk, 42.50.-p, 03.65.Bz

Quantum communication, such as quantum key distribution and quantum teleportation, is hampered by the difficulty to generate maximally entangled states between distant nodes [1]. Due to loss and decoherence, in reality we can only generate partially entangled states between distant sides [2]. Entanglement purification techniques are needed to concentrate maximally entangled states from partially entangled states [3,4]. For qubit systems, efficient entanglement purification protocols have been found [3-5]. But none of these purification schemes have been realized experimentally due to the great difficulty to perform repeated collective operations in realistic quantum communication systems. Thus, it is of interest to consider purification of continuous variable entanglement. The nonlocal Gaussian continuous variable entangled states (i.e., states whose Wigner functions are Gaussians) can be easily generated by transmitting two-mode squeezed light, and this kind of entanglement has been demonstrated in the recent experiment of continuous variable teleportation [6]. As the first choice for performing continuous entanglement purification, one would consider direct extensions of the purification schemes for qubit systems. But till now in these extensions, no entanglement increase has been found for Gaussian continuous entangled states [7]. Thus, the discussion should be extended to a larger class of operations to purify continuous entangled states. Braunstein et al. [8] have proposed a simple error correction scheme for continuous variables. However, it is not clear whether it can be used for purification. In [9] a protocol to increase the entanglement for the special case of pure two-mode squeezed states has been proposed, which is based on conditional photon number subtraction; the efficiency, however, seems to be an obstacle for its practical realization.

In this paper, we present an entanglement purification scheme with the following properties: (i) For pure states it reaches the maximal allowed efficiency in the asymptotic limit (when the number of pairs of modes goes to infinity); (ii) It can be readily extended to distill maximally entangled states from a relevant class of mixed Gaussian states which result from losses in the light transmission. Furthermore, we propose and analyze a scheme to implement this protocol experimentally using high finesse cavities and cross–Kerr nonlinearities. Our purification protocol generates maximally entangled states in finite dimensional Hilbert spaces. The entanglement in the continuous partially entangled state is transformed to the maximally entangled state with a high efficiency. We begin the paper by describing the entanglement purification protocol for pure two mode squeezed states, and then extend the protocol to include mixed Gaussian continuous states, and last describe the physical implementation of the purification protocol.

First assume that we have generated m entangled pairs A_i, B_i $(i = 1, 2, \cdots m)$ between two distant sides A and B. Each pair of modes A_i, B_i are prepared in the two mode squeezed state $|\Psi\rangle_{A_iB_i}$, which in the number basis has the form

$$|\Psi\rangle_{A_iB_i} = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n |n\rangle_{A_i}|n\rangle_{B_i}, \qquad (1)$$

where $\lambda = \tanh(r)$, and r is the squeezing parameter [10]. The entanglement $E\left(|\Psi\rangle_{A_iB_i}\right)$ of the two-component state (1) is uniquely quantified by the von Neumann entropy of the reduced density operator of one-component. The joint state $|\Psi\rangle_{(A_iB_i\}}$ of the m entangled pairs is simply the product of all the $|\Psi\rangle_{A_iB_i}$, which can be rewritten as

$$|\Psi\rangle_{(A_iB_i\}} = \left(1-\lambda^2\right)^{\frac{m}{2}}\sum_{j=0}^{\infty}\lambda^j\sqrt{f_j^{(m)}}\,|j\rangle_{(A_iB_i\}}, \qquad (2)$$

where $(A_iB_i\}$ is abbreviation of the symbol A_1, B_1, A_2, B_2, $\cdots$ and A_m, B_m, and the normalized state $|j\rangle_{(A_iB_i\}}$ is defined as

$$|j\rangle_{(A_iB_i\}} = \frac{1}{\sqrt{f_j^{(m)}}}\sum_{i_1,i_2,\cdots,i_m}^{i_1+i_2+\cdots+i_m=j} |i_1,i_2,\cdots,i_m\rangle_{(A_i\}}\otimes|i_1,i_2,\cdots,i_m\rangle_{(B_i\}}. \qquad (3)$$

The function $f_j^{(m)}$ in Eq. (2) and (3) is given by $f_j^{(m)} = \frac{(j+m-1)!}{j!(m-1)!}$. To concentrate entanglement of these m entangled pairs, we perform a QND measurement of

*Email: luming.duan@uibk.ac.at

the total excitation number $n_{A_1}+n_{A_2}+\cdots+n_{A_m}$ on the A side (we will describe later how to implement this measurement experimentally). The QND measurement projects the state $|\Psi\rangle_{(A_iB_i)}$ onto a two-party maximally entangled state $|j\rangle_{(A_iB_i)}$ with probability $p_j=\left(1-\lambda^2\right)^m\lambda^{2j}f_j^{(m)}$.The entanglement of the outcome state $|j\rangle_{(A_iB_i)}$ is given by $E\left(|j\rangle_{(A_iB_i)}\right)=\log\left(f_j^{(m)}\right)$. The quantity $\Gamma_j=E\left(|j\rangle_{(A_iB_i)}\right)/E\left(|\Psi\rangle_{A_iB_i}\right)$ defines the entanglement increase ratio, and if $\Gamma_j>1$, we get a more entangled state. Even with a small number m, the probability getting a more entangled state is quite high. It can be easily proven that if m goes to infinity, with unit probability we would get a maximally entangled state with entanglement $mE\left(|\Psi\rangle_{A_iB_i}\right)$. This ensures that this method is optimal in this limit, analogous to the purification protocol presented in [3] for the qubit case. For any finite number of entangled pairs, the present purification protocol is more efficient than that in [3], since it takes advantage of the special relations between the coefficients in the two-mode squeezed state.

An interesting feature of this entanglement purification protocol is that for any measurement outcome $j\neq 0$, we always get a useful maximally entangled state in some finite Hilbert space, though the entanglement of the outcome state $|j\rangle_{(A_iB_i)}$ does not necessarily exceed that of the original state $|\Psi\rangle_{A_iB_i}$ if j is small. It is also interesting to note that a small alternation of this scheme provides a useful method for preparing GHZ–like states in high dimensional Hilbert spaces [11]. The key point is that the modes B_i need not be at the same side in the protocol. Assume we have two entangled pairs B,A_1 and A_2,C distributed at three sides B, A, C, with each pair being prepared in the state (1). Then a local QND measurement of the modes A_1,A_2 at the A side with the outcome $j\neq 0$ generates a three-party GHZ state in the $(j+1)$-dimensional Hilbert space. Obviously, if we have m entangled pairs, we can generate a $(m+1)$-party GHZ state using this method.

In reality, the light transmission will be unavoidably subjected to loss, and then we will not start from an ideal two mode squeezed state, but instead from a mixed state described by the following master equation

$$\dot{\rho}=-i\left(H_{\rm eff}\rho-\rho H_{\rm eff}^{\dagger}\right)+\sum_{i=1}^{m}\left(\eta_A a_{A_i}\rho a_{A_i}^{\dagger}+\eta_B a_{B_i}\rho a_{B_i}^{\dagger}\right) \tag{4}$$

where ρ is the density operator of the m entangled pairs with $\rho(0)=|\Psi\rangle_{(A_iB_i)}\langle\Psi|$, the ideal two mode squeezed state, and the effective Hamiltonian

$$H_{\rm eff}=-i\sum_{i=1}^{m}\left(\frac{\eta_A}{2}a_{A_i}^{\dagger}a_{A_i}+\frac{\eta_B}{2}a_{B_i}^{\dagger}a_{B_i}\right). \tag{5}$$

In Eqs. (4) and (5), a_{α_i} denotes the annihilation operator of the mode α_i ($\alpha=A$ or B), and we have assumed that the damping rates η_A and η_B are the same for all the m entangled pairs based on symmetry considerations, but η_A and η_B may be different to each other.

In many practical cases, it is reasonable to assume that the light transmission noise is small. Let τ denote the transmission time, then $\eta_A\tau$ and $\eta_B\tau$ are small factors. In the language of quantum trajectories [10], to the first order of $\eta_A\tau$ and $\eta_B\tau$, the final state of the m entangled pairs is either $\left|\Psi^{(0)}\right\rangle_{(A_iB_i)}\propto e^{-iH_{\rm eff}\tau}|\Psi\rangle_{(A_iB_i)}$ with no quantum jumps occurred, or $\left|\Psi^{(\alpha_i)}\right\rangle_{(A_iB_i)}\propto\sqrt{\eta_\alpha\tau}a_{\alpha_i}|\Psi\rangle_{(A_iB_i)}$ with a jump occurred in the α_i channel ($\alpha=A,B$ and $i=1,2,\cdots,m$). The final density operator is a mixture of all these possible states. To purify entanglement from the mixed state, we perform QND measurements of the total excitation number on both sides A and B, and the measurement results are denoted by j_A and j_B, respectively. We then compare j_A and j_B through classical communication, and keep the outcome state if and only if $j_A=j_B$. Let $P_A^{(j)}$ and $P_B^{(j)}$ denote the projections onto the eigenspaces of the corresponding total number operators $\sum_{i=1}^{m}a_{A_i}^{\dagger}a_{A_i}$ and $\sum_{i=1}^{m}a_{B_i}^{\dagger}a_{B_i}$ with eigenvalue j, respectively. It is easy to show that

$$\begin{aligned}P_A^{(j)}P_B^{(j)}\left|\Psi^{(0)}\right\rangle_{(A_iB_i)}&=|j\rangle_{(A_iB_i)},\\ P_A^{(j)}P_B^{(j)}\left|\Psi^{(\alpha_i)}\right\rangle_{(A_iB_i)}&=0.\end{aligned} \tag{6}$$

So if $j_A=j_B=j$, the outcome state is the maximally entangled state $|j\rangle_{(A_iB_i)}$ with entanglement $\log\left(f_j^{(m)}\right)$. The probability to get the state $|j\rangle_{(A_iB_i)}$ is now given by $p_j^{'}=\left(1-\lambda^2\right)^m\lambda^{2j}f_j^{(m)}e^{-(\eta_A+\eta_B)\tau j}$. It should be noted that the projection operators $P_A^{(j)}P_B^{(j)}$ cannot eliminate the states obtained from the initial state $|\Psi\rangle_{(A_iB_i)}$ by a quantum jump on each side A and B. The total probability for occur of this kind of quantum jumps is proportional to $m^2\overline{n}^2\eta_A\eta_B\tau^2$. So the condition for small transmission noise requires $m^2\overline{n}^2\eta_A\eta_B\tau^2\ll 1$, where $\overline{n}=\sinh^2(r)$ is the mean photon for a single mode.

In the purification for mixed entanglement, we need classical communication (CC) to confirm that the measurement outcomes of the two sides are the same, and during this CC, we implicitly assume that the storage noise for the modes is negligible. In fact, that the storage noise is much smaller than the transmission noise is a common assumption taken in all the entanglement purification schemes which need the help of repeated CCs [4,5]. If we also make this assumption for continuous variable systems, there exists another simple configuration for the purification protocol to work. We put the generation setup for two-mode squeezed states on the A side. After state generation, we keep the modes A_i

on side A with a very small storage loss rate η_A, and at the same time the modes Bi are transmitted to the distant side B with a loss rate $\eta_B \gg \eta_A$. We call this a configuration with an asymmetric transmission noise. In this configuration, the purification protocol is exactly the same as that described in the above paragraph. We note that the component in the final mixed density operator which is kept by the projection $P_A^{(j)} P_B^{(j)}$ should be subjected to the same times of quantum jumps on each side A and B. We want this component to be a maximally entangled state. This requires that the total probability for sides A and B to subject to the same nonzero times of quantum jumps should be very small. This total probability is always smaller than $\overline{n}\eta_A\tau$, despite how large the damping rate η_B is. So the working condition of the purification protocol in the asymmetric transmission noise configuration is given by $\overline{n}\eta_A\tau \ll 1$. The loss rate η_B can be large. The probability to get the maximally entangled state $|j\rangle_{(A_iB_i)}$ is still given by $p_j^{'} = \left(1-\lambda^2\right)^m \lambda^{2j} f_j^{(m)} e^{-(\eta_A+\eta_B)\tau j}$.

For continuous variable systems the assumption of storage with a very small loss rate is typically unrealistic. If this is the case, then we can use the following simple method to circumvent the storage problem. Note that the purpose to distill maximally entangled states is to directly apply them in some quantum communication protocols, such as in quantum cryptography or in quantum teleportation. So we can modify the above purification protocol by the following procedure: right after the state generation, we take a QND measurement of the total excitation number on side A and get a measurement result j_A. Then we do not store the outcome state on side A, but immediately use it (e.g., perform the corresponding measurement as required by a quantum cryptography protocol [12]). During this process, the modes B_i are being sent to the distant side B, and when they arrive, we take another QND measurement of the modes B_i and get a outcome j_B. The resulting state on side B can be directly used (for quantum cryptography for instance) if $j_A = j_B$, and discarded otherwise. By this method, we formally get maximally entangled states through posterior confirmation, and at the same time we need not store the modes on both sides.

To experimentally implement the above purification scheme, we need first generate Gaussian continuous entangled states between two distant sides, and then perform a local QND measurement of the total excitation number of several entangled pairs. Here we propose a promising experimental scheme, which uses high finesse optical cavity to carry continuous entangled states and cavity enhanced cross Kerr interactions to realize the local QND measurement. It is possible to generate Gaussian continuous entangled states between two distant cavities [13]. We can transmit and then couple the two output lights of the nondegenerate optical parametric amplifier to distant high finesse cavities. The steady state of the cavities is just a Gaussian continuous entangled state described by the solution of Eq. (4) after taking into account of the propagation loss [14]. The difficult part is to perform a QND measurement of the total photon number contained in several local cavities. We use the setup depicted in Fig. 1 to attain this goal. (For convenience, we use the two-cavity measurement as an example to illustrate the method. Extension of the measurement method to multi-cavity cases is straightforward.)

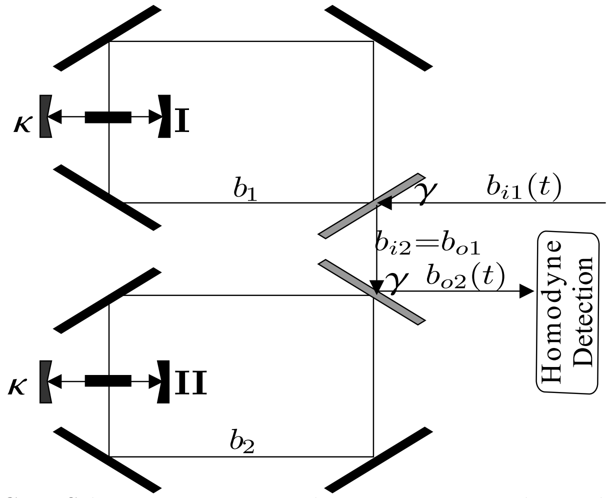

FIG. 1. Schematic experimental setup to measure the total photon number $n_1 + n_2$ contained in the cavities I and II. The cavities I and II, each with a small damping rate κ and with a cross Kerr medium inside, are put respectively in a bigger ring cavity. The ring cavities with the damping rate γ are used to enhance the cross Kerr interactions. A strong cotinuous coherent driving light $b_{i1}(t)$ is incident on the first ring cavity, whose output b_{o1} is directed to the second ring cavity. The output $b_{o2}(t)$ of the second ring cavity is continuously observed through a homodyne detection.

The measurement model depicted in Fig. 1 is an example of the cascaded quantum system [10]. The incident light b_{i1} can be expressed as $b_{i1} = b_{i1}^{'} + g\sqrt{\gamma}$, where $g\sqrt{\gamma}$ (g is a large dimensionless factor) is a constant driving field, and $b_{i1}^{'}$ is the standard vacuum white noise, satisfying $\left\langle b_{i1}^{'\dagger}(t)\, b_{i1}^{'}\left(t^{'}\right)\right\rangle = 0$ and $\left\langle b_{i1}^{'}(t)\, b_{i1}^{'\dagger}\left(t^{'}\right)\right\rangle = \delta\left(t-t^{'}\right)$. The Hamiltonian for the Kerr medium is assumed to be $H_i = \hbar\chi n_i b_i^\dagger b_i$, ($i = 1$ or 2) ,where b_i is the annihilation operator for the ring cavity mode, and χ is the cross-phase modulation coefficient. The self-phase modulation can be made much smaller than the cross phase modulation with some resonance conditions for the Kerr medium, and thus is negligible [15,16]. In the frame rotating at the optical frequencies, the Langevin equations describing the dynamics in the two ring cavities have the form

$$\dot{b_1} = -i\chi n_1 b_1 - \frac{\gamma}{2} b_1 - \sqrt{\gamma} b_{i1}^{'} - g\gamma,$$

$$\dot{b}_2 = -i\chi n_2 b_2 - \frac{\gamma}{2} b_2 - \sqrt{\gamma} b_{i2}, \tag{7}$$

with the boundary conditions (see Fig. 1) $b_{i2} = b_{o1} = b_{i1}^{'} + g\sqrt{\gamma} + \sqrt{\gamma} b_1$ and $b_{o2} = b_{i2} + \sqrt{\gamma} b_2$. In the realistic case $\gamma \gg \chi \langle n_i \rangle$, $(i = 1, 2)$, we can adiabatically eliminate the cavity modes b_i, and express the final output b_{o2} of the second ring cavity as an operator function of the observable $n_1 + n_2$. The experimentally measured quantity is the integration of the homodyne photon current over the measurement time T. Choosing the phase of the driving field so that $g = i|g|$, the measured observable corresponds to the operator

$$\begin{aligned} X_T &= \frac{1}{T} \int_0^T \frac{1}{\sqrt{2}} \left[b_{o2}(t) + b_{o2}^{\dagger}(t) \right] dt \\ &\approx \frac{4\sqrt{2}\,|g|\,\chi}{\sqrt{\gamma}} (n_1 + n_2) + \frac{1}{\sqrt{T}} X_T^{(b)}, \end{aligned} \tag{8}$$

where $X_T^{(b)} = \frac{1}{\sqrt{2}} \left(b_T + b_T^{\dagger} \right)$, and b_T, satisfying $\left[b_T, b_T^{\dagger} \right] = 1$, is defined by $b_T = \frac{1}{\sqrt{T}} \int_0^T b_{i1}^{'}(t)\, dt$. Equation (8) assumes $\gamma \gg \chi \langle n_i \rangle$ and $e^{-\gamma T} \ll 1$. There are two different contributions in Eq. (8). The first term represents the signal, which is proportional to $n_1 + n_2$, and the second term is the vacuum noise. The distinguishability of this measurement is given by $\delta n = \frac{\sqrt{\gamma}}{8|g|\chi\sqrt{T}}$. If $\delta n < 1$, i.e., if the measuring time $T > \frac{\gamma}{64|g|^2\chi^2}$, we effectively perform a measurement of $n_1 + n_2$; and if T is also smaller than $\frac{1}{\kappa\langle n_i \rangle}$, the photon loss in the cavities I and II during the measurement is negligible. So the setup gives an effective QND measurement of the total photon number operator $n_1 + n_2$ under the condition

$$\frac{\gamma}{64\,|g|^2\,\chi^2} < T < \frac{1}{\kappa\,\langle n_i \rangle}. \tag{9}$$

This condition seems to be feasible with the present technology. For example, if we assume the cross Kerr interaction is provided by the resonantly enhanced Kerr nonlinearity as considered and demonstrated in [15,16], the Kerr coefficient $\chi/2\pi \sim 0.1MHz$ would be obtainable [17]. We can choose the decay rates $\kappa/2\pi \sim 4MHz$ and $\gamma/2\pi \sim 100MHz$, and let the dimensionless factor $g \sim 100$ (for a cavity with cross area $S \sim 0.5 \times 10^{-4} cm^2$, $g \sim 100$ corresponds a coherent driving light with intensity about $40mWcm^{-2}$). The mean photon number $\langle n_1 \rangle = \langle n_2 \rangle = \sinh^2(r) \sim 1.4$ for a practical squeezing parameter $r \sim 1.0$. With the above parameters, Eq. (9) can be easily satisfied if we choose the measuring time $T \sim 8ns$. More favorable values for the parameters are certainly possible.

To bring the above proposal into a real experiment, there are several imperfect effects which should be considered. These imperfections include phase instability of the driving field, unbalance between the two ring cavities, light absorption of the Kerr media and the mirrors, self phase modulation effects, light transmission loss between the ring cavities, and inefficiency of the detectors. To realize a QND measurement, the imperfections should be small enough. We have deduced quantitative requirements for all the imperfections listed above [18]. With the parameters given in the above paragraph, all these requirements can be met experimentally.

We thank P. Grangier and S. Parkins for discussions. This work was supported by the Austrian Science Foundation, by the European TMR network Quantum Information, and by the Institute for Quantum Information.

[1] C. H. Bennett, Phys. Today 48 (10), 24 (1995)
[2] J. I. Cirac et al., Phys. Rev. Lett. 78, 3221 (1997); S. J. Enk, J. I. Cirac, and P. Zoller, Science 279, 205 (1998).
[3] C. H. Bennett et al., Phys. Rev. A 53, 2046 (1996).
[4] C. H. Bennett et al., Phys. Rev. Lett. 76, 722 (1996).
[5] C. H. Bennett et al., Phys. Rev. A 54, 3824 (1996).
[6] A. Furusawa, et al., Science 282, 706 (1998).
[7] S. Parker, S. Bose, and M. B. Plenio, e-print quant-ph/9906098.
[8] S. L. Braunstein, Nature 394, 47 (1998); Phys. Rev. Lett. 80, 4084 (1998); S. Lloyd and J. J.-E Slotine, Phys. Rev. Lett. 80, 4088 (1998).
[9] T. Opatrny, G. Kurizki, and D.-G., Welsch, e-print quant-ph/9907048.
[10] C. W. Gardiner and P. Zoller, *Quantum Noise* (Springer–Verlag, Berlin, 1999).
[11] D. Greenberger et al., Am. J. Phys. 58, 1131 (1990); J. W. Pan et al., Nature (to be published); G. M. D'Ariano et al. e-print quant-ph/9906067.
[12] M. Hillery, e-print quant-ph/9909006.
[13] N. Ph. Georgiades et al., Phys. Rev. Lett. 75, 3426 (1995).
[14] A. S. Parkins and H. J. Kimble, e-print quant-ph/9907048.
[15] A. Imamoglu at al., Phys. Rev. Lett. 79, 1467 (1997); 81, 2836 (1998).
[16] L. V. Hau, et al., Nature 397, 594 (1999).
[17] In fact, Ref. [15] considered a configuation, yielding a Kerr coefficient $\chi \sim 100MHz$, to realize a single-photon turnstile device. But the estimation there puts a stringent limit on the required cavity parameters [K. M. Gheri *et al.*, Phys. Rev. A 60, R2673, 1999]. We take a much more moderate estimation of the relavent parameters and find $\chi/2\pi \sim 0.1MHz$ is obtainable. This value of the Kerr coefficient is large enough for performing the QND measurement, though it is certainly not enough for realizing a single-photon turnstile device.
[18] L. M. Duan et. al., (to be published).